

EMBEDDED SURFACES AND ALMOST COMPLEX STRUCTURES

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ABSTRACT. In this paper, we prove necessary and sufficient conditions for a smooth surface in a smooth 4-manifold X to be pseudoholomorphic with respect to an almost complex structure on X . In particular, this provides a systematic approach to the construction of pseudoholomorphic curves that do not minimize the genus in their homology class.

1. INTRODUCTION AND SUMMARY OF RESULTS

Let X be a closed differentiable and connected 4-manifold with an orientation and $\Sigma \subset X$ a connected oriented surface. An **almost complex structure** J on X is a reduction of the structure group $GL^+(4)$ of TX to the group $GL(2, \mathbb{C})$, or, in other words, a section J of $End(TX)$ with $J^2 = -1$ that preserves the orientation, so that TX carries the structure of a complex vector bundle. The surface Σ is called a **pseudoholomorphic curve** if the tangent bundle of Σ is preserved by J (note that in this case, the almost complex structure on X induces a complex structure on Σ). The question that shall be treated on the following pages is : Given a surface Σ , is there an almost complex structure J on X such that Σ is a pseudoholomorphic curve with respect to J ?

First, recall that an almost complex structure J has a first Chern class $c_1(J) \in H^2(X; \mathbb{Z})$ (this is just the first Chern class of TX considered as a complex vector bundle) and that this class has the properties

1. $c_1(J)^2 = 2\chi + 3\tau$
2. $c_1(J) \equiv w_2 \pmod{2}$

where χ denotes the Euler characteristic and τ the signature of the intersection form of X . A class with property 2 is called a **characteristic class** on X . If the homology of X does not contain 2-torsion, then these classes can be characterized in terms of the intersection form Q of X : a class $c \in H^2(X; \mathbb{Z})$ is characteristic if and only if $Q(x, x) \equiv Q(x, c) \pmod{2}$ for all $x \in H^2(X; \mathbb{Z})$. If there is 2-torsion, one part of this statement is still true: if c is characteristic, then $Q(x, c) \equiv Q(x, x) \pmod{2}$ for every x . Conversely, if a class c fulfills $Q(x, c) \equiv Q(x, x) \pmod{2}$ for all x , then there is a torsion class a such that $c + a \equiv w_2(X) \pmod{2}$. It is a classical result of Whitney that there are characteristic classes on any 4-manifold ([W]).

Furthermore, it is well known that in turn every class in $H^2(X; \mathbb{Z})$ fulfilling the conditions 1,2 can be realized as the first Chern class of an almost complex structure. So there is an almost complex structure on X if and only if there is a class in $H^2(X; \mathbb{Z})$ that fulfills the conditions above (this is a result of Wu, see [HH]), in fact every such class is the first Chern class of an almost complex structure.

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Consideration of the intersection form easily leads to the conclusion that if the intersection form of X is indefinite, there is an almost complex structure on X if and only if $b_1 + b_2^+$ is odd, where b_2^+ denotes the maximal dimension of a subspace of $H^2(X, \mathbb{R})$ on which the intersection form is positive definite.

Definition 1. Let G be a finitely generated abelian group and $g \in G$. Let $T \subset G$ be the torsion subgroup of G .

1. If G is free abelian and $g \neq 0$, the **divisibility** of g in G is defined to be the largest positive integer d with the property that there is an $x \in G$ with $g = dx$. The divisibility of $0 \in G$ is defined to be zero.
2. For arbitrary G , the divisibility $d(g)$ of $g \in G$ is the divisibility of g (more precisely, the residue class of g) in the free abelian group G/T . The divisibility is defined to be zero if and only if $g \in T$.

Remark 1. Clearly the image of the homomorphism $\text{Hom}(G; \mathbb{Z}) \rightarrow \mathbb{Z}$, given by evaluation on g , is just $d(g)\mathbb{Z}$. From this, we see that the divisibility of $k \cdot g$ for $g \in G$, $k \in \mathbb{Z}$ is $\pm k$ times the divisibility of g .

Definition 2. Let (Γ, Q) be a lattice (i.e. Γ is a free abelian group of finite rank and Q a unimodular symmetric bilinear form on Γ). For $\gamma \in \Gamma$ with divisibility $d = d(\gamma)$ define $k(\gamma) \in \mathbb{Z}_d$ as follows: Choose a characteristic class $c \in \Gamma$, i.e. $Q(c, x) \equiv Q(x, x) \pmod{2}$ for every $x \in \Gamma$, and set

$$k(\gamma) := 1 + \frac{1}{2}(Q(\gamma, \gamma) - Q(c, \gamma)) \pmod{d}$$

The residue class $k(\gamma)$ is independent of the choice of c : if c' is another characteristic class, then c and c' differ by a multiple of 2, so the terms in the bracket differ by a multiple of $2d$, according to remark 1, and this does not affect $k(\gamma)$. If $\gamma = 0$, then $k(\gamma) = 1 \in \mathbb{Z}$.

Definition 3. For a closed connected and oriented surface $\Sigma \subset X$ let $k(\Sigma) = k([\Sigma])$ with respect to the lattice defined by the intersection form on the free group $H_2(X; \mathbb{Z})/\text{Tor } H_2(X; \mathbb{Z})$, where $[\Sigma]$ denotes the homology class of Σ .

Since cup products are not altered by adding torsion classes to one of the factors, we could as well have defined $k([\Sigma])$ by

$$k([\Sigma]) = 1 + \frac{1}{2}(\Sigma \cdot \Sigma - c \cdot \Sigma) \pmod{d}$$

where d denotes the divisibility of $[\Sigma]$ in $H_2(X; \mathbb{Z})$ and c is any characteristic class on X , i.e. $c \equiv w_2(X) \pmod{2}$. We will use the notation $d(\Sigma)$ for the divisibility of the class $[\Sigma]$.

If there is an almost complex structure J turning Σ into a pseudoholomorphic curve, then the adjunction formula

$$g(\Sigma) = 1 + \frac{1}{2}(\Sigma \cdot \Sigma - c_1(J) \cdot \Sigma)$$

holds, and $c_1(J)$ is characteristic, so we have the congruence

$$g(\Sigma) \equiv k(\Sigma) \pmod{d}.$$

It turns out that this necessary condition is in fact sufficient for the existence of such a J if the intersection form of X is strictly indefinite (i.e. $\min\{b_2^+, b_2^-\} \geq 2$):

Theorem 1. *Let X be a connected oriented closed and differentiable 4-manifold and $\Sigma \subset X$ a closed connected and oriented surface with divisibility d . Suppose $\min\{b_2^+, b_2^-\} \geq 2$ and $b_1 + b_2^+ \equiv 1 \pmod{2}$. Then there is an almost complex structure J on X such that the surface Σ is pseudoholomorphic with respect to J if and only if $g(\Sigma) \equiv k(\Sigma) \pmod{d}$.*

Note that this condition is in particular fulfilled when the class of Σ has divisibility one, so any such surface is pseudoholomorphic with respect to an almost complex structure on X . In addition, if $[\Sigma]$ is not a torsion class, this condition is “cyclic” : we can attach handles that do not change the homology class of Σ - and hence do not alter $k(\Sigma)$ - but raise the genus until the condition of the Theorem is fulfilled. In this way we even can produce a surface Σ' homologous to Σ that is pseudoholomorphic with respect to an almost complex structure on X , but whose genus is arbitrarily large:

Corollary 1. *Let X be a closed connected and oriented smooth 4-manifold with $b_1 + b_2^+ \equiv 1 \pmod{2}$ and $\min\{b_2^+, b_2^-\} \geq 2$, and let Σ be a surface in X such that $[\Sigma]$ is not a torsion class. Let $m \in \mathbb{N}$ be any natural number. Then there is an almost complex structure J on X and a pseudoholomorphic curve Σ' homologous to Σ with $g(\Sigma') \geq g(\Sigma) + m$.*

This Corollary provides a large number of pseudoholomorphic curves that do not minimize the genus in their homology class. Other examples for this have been given by Kotschick (unpublished) and in a paper by Mikhalkin ([Mi]). Although the case $X = \mathbb{C}P^2$ is not covered by the Corollary, this can also occur on $\mathbb{C}P^2$, an example for this is the curve in the statement of Proposition 4.

The next three Propositions show that the condition $\min\{b_2^+, b_2^-\} \geq 2$ is really necessary, if it is dropped, the Theorem is no longer true:

Proposition 1. *If X is a rational complex surface, there is a surface Σ in X with $g(\Sigma) \equiv k(\Sigma) \pmod{d(\Sigma)}$ that is not pseudoholomorphic with respect to any almost complex structure on X .*

Proposition 2. *Let X be a 4-manifold with definite intersection form. Then there is a surface Σ in X with $k(\Sigma) \equiv g(\Sigma) \pmod{d(\Sigma)}$ that is not pseudoholomorphic with respect to any almost complex structure on X .*

Proposition 3. *Let Q be a unimodular symmetric bilinear form over \mathbb{Z} fulfilling $\min\{b^+, b^-\} \leq 1$ that can be realized as the intersection form of a smooth 4-manifold. In the case that Q is indefinite and even assume that the signature of Q is non-negative. Then there is a closed oriented 4-manifold X having Q as intersection form and a closed oriented and connected surface $\Sigma \subset X$ such that $b_1(X) + b_2^+(X) \equiv 1 \pmod{2}$ and $g(\Sigma) \equiv k(\Sigma) \pmod{d(\Sigma)}$, but Σ is not pseudoholomorphic with respect to any almost complex structure on X .*

Note that all odd forms and all even intersection forms of smooth 4-manifolds that have no 2-torsion in their homology ([D]) or are spin ([Fu]) are covered by this Proposition.

Finally, there is a simple example for the case $X = \mathbb{C}P^2$. The class $-1 \in H_2(\mathbb{C}P^2; \mathbb{Z})$ can be represented by a sphere - just take the complex line with the orientation reversed -, hence the minimal genus for this class is 0. The following Proposition therefore provides another example that a pseudoholomorphic curve does not always minimize the genus in its homology class:

Proposition 4. *There is a surface with genus 3, representing minus the generator of $H_2(\mathbb{CP}^2; \mathbb{Z})$, that is pseudoholomorphic with respect to an almost complex structure homotopic to the canonical one.*

2. PROOFS OF THEOREM 1 AND PROPOSITION 4

For the proofs, we need two Lemmas, the first of them being a topological Lemma, whereas the second one is purely algebraic:

Lemma 1. *If there is a class $c \in H^2(X; \mathbb{Z})$ with the following properties*

1. $c^2 = 2\chi + 3\tau$
2. $c \equiv w_2 \pmod{2}$
3. $\langle c, [\Sigma] \rangle = 2 - 2g(\Sigma) + \Sigma \cdot \Sigma$,

then there is an almost complex structure J such that the surface Σ is pseudoholomorphic with respect to J and $c_1(J) = c$ (here τ denotes the signature of X).

Proof. By the result of Wu mentioned earlier (see [HH]), there is an almost complex structure J_0 on X with $c_1(J_0) = c$. Introduce a Riemannian metric g on X compatible with J_0 , i.e. the endomorphism J is isometric on the fibers of TX with respect to g . Then the almost complex structures compatible with g can be identified with the reductions of the structure group $SO(4)$ to $U(2)$, i.e. with sections in the bundle $\Theta := P_{SO(4)}/U(2)$ having fiber $SO(4)/U(2) = S^2$. For the restriction of the tangent bundle to Σ , we have a decomposition $TX|_\Sigma = N \oplus T\Sigma$, where N denotes the normal bundle of Σ . By introducing metrics on these two bundles, their structure group can be reduced to $SO(2)$. Since $SO(2) \times SO(2) \subset U(2)$, we have an almost complex structure on $TX|_\Sigma$ turning this decomposition into a direct sum of complex vector bundles. This almost complex structure can be extended to an almost complex structure J on the disk bundle DN (that is identified with a tubular neighborhood of Σ). Clearly, Σ is a pseudoholomorphic curve in DN with respect to J . We now have to show that J can be extended over X to an almost complex structure homotopic to J_0 as a section of Θ , then $c_1(J) = c_1(J_0) = c$, and the Lemma is proved.

The second cohomology $H^2(DN; \mathbb{Z})$ is \mathbb{Z} , generated by the fundamental class $[\Sigma]$ (more exactly, by its pullback via the projection $DN \rightarrow \Sigma$). Let s respectively s_0 denote the sections of Θ on DN given by J and J_0 . Note that J_0 defines an extension of s_0 to X . Let $c_1 \in H^2(DN; \mathbb{Z})$ denote the first Chern class of J . By definition of J , we have a decomposition $(TX|_\Sigma, J) = N \oplus T\Sigma$ of complex vector bundles. Taking the first Chern class on both sides yields the adjunction equality $\langle c_1, \Sigma \rangle = 2 - 2g + \Sigma \cdot \Sigma$. But by assumption 3, the same is true for $c = c_1(J_0)$, hence $c_1 = c$ in $H^2(DN; \mathbb{Z})$. A short calculation, using the exact homotopy sequence of the fibration

$$S^2 \rightarrow BU(2) \rightarrow BSO(4),$$

yields that $\pi_2(S^2) \rightarrow \pi_2(BU(2))$ is the multiplication by 2, and this shows that for the primary difference $p \in H^2(DN; \mathbb{Z})$ between s and s_0 as sections $DN \rightarrow \Theta$, we have the equality $2p = c_1 - c = 0$. Since the homology of DN is torsion free, this implies $p = 0$, and since $H^3(DN; \mathbb{Z}) = H^4(DN; \mathbb{Z}) = 0$, there are no higher obstructions, hence the sections s and s_0 are homotopic on DN . Using the homotopy extension property we can conclude that there is an extension of s to X homotopic to s_0 , and this proves the assertion. \square

Lemma 2. *Let (Γ, Q) be a lattice with $\min\{b^+, b^-\} \geq 2$, let $\gamma \in \Gamma$ be a vector with divisibility d , h an integer with $h \equiv \tau(Q) \pmod{8}$, where $\tau(Q)$ denotes the signature of Q , and g be a natural number with $g \equiv k(\gamma) \pmod{d}$. Then there is a $c \in \Gamma$ with*

1. *c is characteristic, i.e. $Q(c, x) \equiv Q(x, x) \pmod{2}$ for every $x \in \Gamma$,*
2. *$Q(c, c) = h$ and*
3. *$Q(c, \gamma) = 2 - 2g + Q(\gamma, \gamma)$.*

Proof. According to the classification Theorem of Hasse-Minkowski (see [MH]), we can choose a basis (e_1, \dots, e_n) such that with respect to this basis, Q is described by the matrix

$$\begin{pmatrix} \boxed{H} & & \\ & \boxed{H} & \\ & & \boxed{A} \end{pmatrix}$$

where $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ denotes the hyperbolic form, and A is diagonal if Q is odd, or of the type mE_8 with some $m \in \mathbb{Z}$ in the case that Q is even. If $\gamma = 0$, the condition on g reads $g = 1$, and any characteristic c with $Q(c, c) = h$ will do the job (it is easy to see that such a c exists). Now assume $\gamma \neq 0$. Let $\gamma = dp$ with a $p \in \Gamma$ having divisibility one.

Case 1: Q is even. Then p must be ordinary (i.e. not characteristic), because Q is unimodular and - according to the characterisation of the divisibility given in Remark 1 - therefore there is an $x \in \Gamma$ with $Q(x, p) = 1$. Using a result of Wall ([W1]) concerning the group of automorphisms of Q , we can assume $p = (k, 1, 0, \dots, 0)$ with some $k \in \mathbb{Z}$ (Wall's Theorem asserts that there is an automorphism that maps p to some vector of this type, after a change of the basis, we can assume that p has this special form). Let $c_0 \in \langle e_3, \dots, e_n \rangle$ be some characteristic vector with $Q(c_0, c_0) = h$ (it is easy to see that such a c_0 exists, using the Hasse-Minkowski classification applied to $H \oplus A$). The assumption on g implies that the difference between $Q(c_0, \gamma)$ and $2 - 2g + Q(\gamma, \gamma)$ is a multiple of $2d$, say $2da$ with $a \in \mathbb{Z}$. Let $c = c_0 + 2ae_1$. Then $Q(c, c) = Q(c_0, c_0) = h$ and $Q(c, \gamma) = Q(c_0, \gamma) + 2ad = 2g + Q(\gamma, \gamma)$.

Case 2: Q is odd:

- a) p is ordinary: Then, again using the result of Wall, we can assume that $p = (k, 1, p')$ with $p' \in \langle e_3, \dots, e_n \rangle$, and the same arguments as in Case 1 apply.
- b) p is characteristic: Since Q is odd, the same must be true for A , in particular, $n \geq 5$. In this case, the standard form for p is $p = (0, 0, 2k, 2, 1, \dots, 1)$ with some $k \in \mathbb{Z}$, because this vector is characteristic, has divisibility one and square $8k + \tau(Q)$, and $Q(p, p) \equiv \tau(Q) \pmod{8}$. Choose any characteristic vector $c_0 \in \langle e_3, \dots, e_n \rangle$. Then the difference $Q(c_0, c_0) - (2 - 2g + Q(\gamma, \gamma))$ is divisible by $2d$ (this follows from the assumption on g and the definition of $k(\gamma)$). Therefore we can choose $a \in \mathbb{Z}$ such that $c_1 := c_0 + 2ae_5$ has the properties 1 and 3 (observe $Q(e_5, p) = \pm 1$).

The difference between $Q(c_1, c_1)$ and h is now a multiple of 8, say $8b$, $b \in \mathbb{Z}$, and therefore $c = c_1 + 2be_1 + 2e_2$ fulfills all the three conditions. \square

Proof. of Proposition 4: Let J denote the standard almost complex structure on $\mathbb{C}P^2$. with Chern class $c_1(J) = 3$. Let Σ' denote the complex line $\mathbb{C}P^1$ in $\mathbb{C}P^2$ with the orientation reversed, hence $[\Sigma'] = -1$. By attaching handles, we can construct a surface $\Sigma \subset \mathbb{C}P^2$ with genus 3 homologous to Σ' . For this surface, we have

$$\langle c_1(J), [\Sigma] \rangle = -3 = 2 - 2g(\Sigma) + \Sigma \cdot \Sigma,$$

and the assertion follows using the homotopy argument as in the proof of Lemma 1. \square

Proof. of Theorem 1: First assume that Σ is pseudoholomorphic with respect to J . Then we have the adjunction equality $c_1(J) \cdot \Sigma = 2 - 2g + \Sigma \cdot \Sigma$, and this implies $g(\Sigma) \equiv k(\Sigma) \pmod{d}$. For the converse, let Σ fulfill $g(\Sigma) = k(\Sigma) \pmod{d}$. Let $\Gamma = H^2(X; \mathbb{Z}) / \text{Tor } H^2(X; \mathbb{Z})$ and Q denote the form on Γ defined by the intersection form. Let $\gamma \in \Gamma$ be the residue class of $[\Sigma]$ and $h := 2\chi + 3\tau$. A short calculation shows that the condition $b_1 + b_2^+ \equiv 1 \pmod{2}$ implies that $\chi + \tau$ is divisible by 4 and $h \equiv \tau(Q) \pmod{8}$. According to Lemma 2, there is a $c' \in \Gamma$ with $Q(c', c') = h$, $Q(c', \gamma) = 2 - 2g + Q(\gamma, \gamma)$ and $Q(c', x) \equiv Q(x, x) \pmod{2}$ for all $x \in \Gamma$. Choose a lift $c \in H^2(X; \mathbb{Z})$ of c' such that $c \equiv w_2(X) \pmod{2}$. Then c fulfills the conditions of Lemma 1, and the assertion of the Theorem follows. \square

3. Proof of Propositions 1,2 and 3

Proof. of Proposition 1: A rational surface is diffeomorphic to $S^2 \times S^2$ or to $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$. First, consider the case $X = S^2 \times S^2$. Let Δ denote the diagonal sphere in $S^2 \times S^2$. Whenever $c = (x, y) \in H^2(X; \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}$ is the Chern class of an almost complex structure, we have $c^2 = 2xy = 8$, and x, y are even, therefore we have $c = (2, 2)$ or $c = (-2, -2)$. Now choose a surface Σ homologous to Δ . Observe that, since Δ is a sphere, one can construct such surfaces Σ of any genus by attaching nullhomologous handles to Δ . Then, for any almost complex structure J on X , we have $c_1(J) \cdot \Sigma = \pm 4$ and $\Sigma \cdot \Sigma = 2$. If we choose Σ to have genus 0 or 4, we see that there is an almost complex structure on X that turns Σ into a pseudoholomorphic curve, but if we choose a surface $\Sigma \sim \Delta$ with genus 1, then there is no almost complex structure on X such that Σ is pseudoholomorphic. But on the other hand, the divisibility of Σ clearly is one, so the equality $g(\Sigma) \equiv k(\Sigma) \pmod{d}$ is fulfilled for every value of $g(\Sigma)$.

Now consider the case $X = \mathbb{C}P^2$. Clearly, only the classes 3 and -3 occur as Chern classes of almost complex structures on X . If we construct a surface Σ of genus one, representing the generator of $H^2(X; \mathbb{Z})$, by attaching a handle to $\mathbb{C}P^1$, then, for any almost complex structure J , $c_1(J) \cdot \Sigma = \pm 3$, $\Sigma \cdot \Sigma = 1$, so the adjunction equality will not hold for J . Again, we have $d = 1$, and this provides the required example.

Now we turn to the case $X = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Let Σ be a surface of genus one, representing the class $(1, 0) \in H^2(X; \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}$. Then the divisibility is 1, and we have to prove that there is no almost complex structure such that Σ is pseudoholomorphic. If J is an almost complex structure on X with first Chern class $c = (x, y)$, then $x^2 - y^2 = 8$. If Σ would be pseudoholomorphic with respect to J , this would imply $c \cdot \Sigma = x = 1$, hence $1 - y^2 = 8$, contradiction. This proves that there is no such J .

Finally, to settle the case $X = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ and $k \geq 2$, note that X is diffeomorphic to $(S^2 \times S^2) \# (k-1)\overline{\mathbb{C}P^2}$ ([W2]). Let again $\Delta \subset X$ denote the sphere coming from the diagonal embedding in $S^2 \times S^2$, choose a surface Σ representing the same homology class and let g denote its genus. With respect to the basis of $H^2(X; \mathbb{Z})$ coming from a diffeomorphism $X \cong (S^2 \times S^2) \# (k-1)\overline{\mathbb{C}P^2}$, we have $[\Sigma] = (1, 1, 0, \dots, 0)$, and if J is an almost complex structure with Chern class $c = (x, y, a)$, with $a \in H^2((k-1)\overline{\mathbb{C}P^2})$ and $x, y \in \mathbb{Z}$, then $c^2 = 2xy + a \cdot a = 9 - k$ (note that $a \cdot a \leq 0$, here the dot denotes the cup product in the cohomology of $(k-1)\overline{\mathbb{C}P^2}$). Now $a \cdot a \leq -k + 1$, since $a \equiv w_2((k-1)\overline{\mathbb{C}P^2}) \pmod{2}$ and the intersection form of $(k-1)\overline{\mathbb{C}P^2}$ is standard, and we can conclude

$$2xy = 9 - k - a \cdot a \geq 8.$$

Therefore, as in the example $X = S^2 \times S^2$, we have $xy \geq 4$, and $x, y \equiv 0 \pmod{2}$. Now suppose that Σ is pseudoholomorphic with respect to J . Then we have the adjunction equality $c \cdot \Sigma = x + y = 4 - 2g$. Together with $xy \geq 4$, this implies $g = 0$ or $g \geq 4$. But we can construct Σ by attaching one handle at Δ and therefore realize $g = 1$. Hence this surface is not pseudoholomorphic with respect to any almost complex structure on X , and this completes the proof. \square

Note that the last part of the proof can be applied to every X of the type $(S^2 \times S^2) \# N$, where N has no 2-torsion in its homology and negative definite intersection form (which must be standard, according to Donaldson).

Proof. of Proposition 2: The intersection form Q can be considered as a non-degenerate symmetric bilinear form on the real cohomology $H^2(X; \mathbb{R})$, where the free part of the integral cohomology is lying as a lattice in this real vector space. Choose a class $\gamma \in H^2(X; \mathbb{Z})$ with self-intersection $s = \gamma \cdot \gamma \neq 0$ and divisibility one. First suppose that Q is positive definite. Then, for every class $c \in H^2(X; \mathbb{Z})$, we have the Cauchy-Schwarz inequality $|Q(\gamma, c)|^2 \leq sQ(c, c)$. If c is the Chern class of an almost complex structure J , this implies $|Q(\gamma, c)|^2 \leq s(2\chi + 3b_2)$. If Σ is a representative of γ that is pseudoholomorphic with respect to J , we therefore have $(2 - 2g + s)^2 \leq s(2\chi + 3b_2)$. Note that the number on the right side of this inequality must be non-negative, otherwise there is no almost complex structure on X at all. Hence we see that there is an upper bound for the genus of pseudoholomorphic curves representing γ that does not depend on J , therefore a representative with large genus provides the required example. A similar argument works if Q is negative definite. \square

Proof. of Proposition 3: If Q is definite, then the assertion of the Proposition is covered by Proposition 2. If both b^+ and b^- equal 1, then Q must be the intersection form of $S^2 \times S^2$ or of $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, hence the intersection form of a rational surface. The same is true if $b^+ = 1$ and Q is odd, all these cases are covered by Proposition 1. So the last case that is not covered by any of the preceding examples is the case $b^- = 1$ and $b^+ \geq 2$. In this case, choose a 4-manifold X' with intersection form Q . Let $h(X') = 2\chi(X') + 3\tau(X')$. Choose a class γ in $H^2(X'; \mathbb{Z})$ with divisibility 1 and $-s^2 := Q(\gamma, \gamma) < 0$. Consider the lattice defined by the integral cohomology in the semi-euclidean vector space $H^2(X; \mathbb{R})$. Choose a basis e_1, \dots, e_n of this vector

space such that with respect to this basis, Q is given by the matrix

$$Q = \begin{pmatrix} -1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

where all other entries are zero, and such that $\gamma = se_1$ (more precisely, the free part of γ). Now choose a surface Σ representing γ with arbitrary genus g . Let $X = X' \# k(S^1 \times S^3)$, where k is chosen large, such that $h(X) = 2\chi(X) + 3\tau(X) < -(\frac{2-2g}{-s} + s)^2$ and $b_1(X) + b_2^+(X) \equiv 1 \pmod{2}$ (note that attaching a copy of $S^1 \times S^3$ decreases the Euler characteristic $\chi(X)$ by 2 without changing the second cohomology and the intersection form). If now J would be any almost complex structure on X such that Σ is pseudoholomorphic with respect to J , and $c = \sum_i c_i e_i \in H^2(X; \mathbb{R})$ its (real) first Chern class, then the adjunction equality would imply $c \cdot \Sigma = -sc_1 = 2 - 2g - s^2$, hence

$$Q(c, c) = -c_1^2 + \sum_{i \neq 1} c_i^2 \geq -c_1^2 = -(\frac{2-2g}{-s} + s)^2 > h(X),$$

in contradiction to $c^2 = 2\chi(X) + 3\tau(X) = h(X)$. This proves that Σ can not be pseudoholomorphic for any J , although, since the divisibility of γ is one, the condition $g \equiv k(\gamma) \pmod{d}$ is fulfilled. \square

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